

Friday, Sep 19, Lecture 10: The Spectral Thm

Last time:

A linear operator $T: V \rightarrow V$ is diagonalizable if there is a basis of V of eigenvectors $\{v_1, \dots, v_n\}$:

$$Tv_i = \lambda_i v_i$$

(Equivalently, if there is a change-of-basis matrix S which diagonalizes the matrix $[T]$ in some basis:

$$[T] = S D S^{-1}.)$$

. Inner products, orthogonality, projections

Question: when can a linear operator $T: V \rightarrow V$ be diagonalized?

Def: Orthogonal Projection

Let V be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$, and let U be a subspace of V .

The orthogonal projection $P: V \rightarrow V$ is the linear map satisfying

① $\text{ran}(P) = U$ (P projects onto U)

② $P^2 = P$ (P acts as identity on U : if $u \in U$, $Pu = u$).

③ $\langle Pv, w \rangle = \langle v, Pw \rangle$ for all $v, w \in V$. (P is self-adjoint)

↑ Two special cases of ③, to be discussed:

- $P = P^T$ (real vector spaces)

- $P = P^+$ (complex vector spaces)

Diagonalizing Projections:

- If U has an orthonormal basis $\{u_1, \dots, u_m\}$, it can be computed as before:

$$Pv = \langle u_1, v \rangle u_1 + \dots + \langle u_m, v \rangle u_m.$$

Notice: if $v \in U$, then $Pv = v$
if $v \in U^\perp$, then $Pv = 0$.

This means P is diagonal in the orthonormal basis $\{u_1, \dots, u_m, v_{m+1}, \dots, v_n\}$:

$$[P] = \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & 0 & \dots & 0 \\ & & & & \ddots & \\ & & & & & 0 \end{bmatrix} \begin{matrix} \uparrow \\ \{u\} \\ \uparrow \\ \{u^\perp\} \end{matrix}$$

Let's say more about condition ③.

Def: Transpose and Adjoint:

Let V be a Hilbert space over $\mathbb{R}$ (over $\mathbb{C}$)

and let $A: V \rightarrow V$ be linear. The transpose (adjoint) of A , A^T (A^*) is the linear map

$$\langle Av, w \rangle = \langle v, A^T w \rangle$$

$$\langle Av, w \rangle = \langle v, A^* w \rangle$$

Def:

Let M be an $m \times n$ matrix.

The transpose of M is the matrix M^T

$$M = \begin{pmatrix} m_{ij} \end{pmatrix} \quad M^T = \begin{pmatrix} m_{ji} \end{pmatrix}$$

Ex:

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 0 \end{pmatrix}^T = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 0 \end{pmatrix}$$

$$(1 \ 2 \ 3)^T = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

complex
conjugate
↓

The adjoint of M is the matrix $M^+ = \overline{(M^T)}$

Ex:

$$(i \ 1)^+ = \begin{pmatrix} -i \\ 1 \end{pmatrix}$$

Key property:

Let $V = \mathbb{R}^n$ ($\mathbb{C}^n$) with dot product:

$$\langle v, w \rangle = v \cdot w = \sum_{i=1}^n v_i w_i$$

$$\langle v, w \rangle = \bar{v} \cdot w = \sum_{i=1}^n \bar{v}_i w_i$$

Then

$$\langle Mv, w \rangle = \langle v, M^T w \rangle$$

$$\langle Mv, w \rangle = \langle v, M^+ w \rangle$$

Properties: A, B are $n \times n$ matrices, $a \in \mathbb{R}$ (or $\mathbb{C}$)

• Linearity: $(aA + B)^T = aA^T + B^T$ (Antilinearity: $(aA + B)^* = \bar{a}A^* + B^*$)

• $(AB)^T = B^T A^T$

• $(A^T)^T = A$

• $\det(A^T) = \det(A)$.

Def: Orthogonal, Unitary

We call an $n \times n$ real matrix O orthogonal if

$$OO^T = O^T O = \mathbb{1} \quad \leftarrow \text{identity matrix } \mathbb{1} = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}$$

We call an $n \times n$ complex matrix U unitary if

$$UU^* = U^* U = \mathbb{1}$$

Fact:

The columns of an orthogonal (or unitary) matrix O form an orthonormal basis of $\mathbb{R}^n$ (or $\mathbb{C}^n$):

$$O = \begin{bmatrix} | & & | \\ v_1 & \dots & v_n \\ | & & | \end{bmatrix}$$

and so O represents a change-of-basis between two orthogonal matrices.

They preserve length and angle since for any $v, w \in \mathbb{R}^n$,

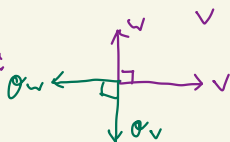
$$\langle Ov, Ow \rangle = \langle v, O^T O w \rangle = \langle v, \mathbb{1} w \rangle = \langle v, w \rangle.$$

Geometrically, this means every orthogonal matrix represents a combination of two fundamental maps:

• Rotation:



• Reflection:



Spectral Theorem

Let $A = A^T$. A is "symmetric".

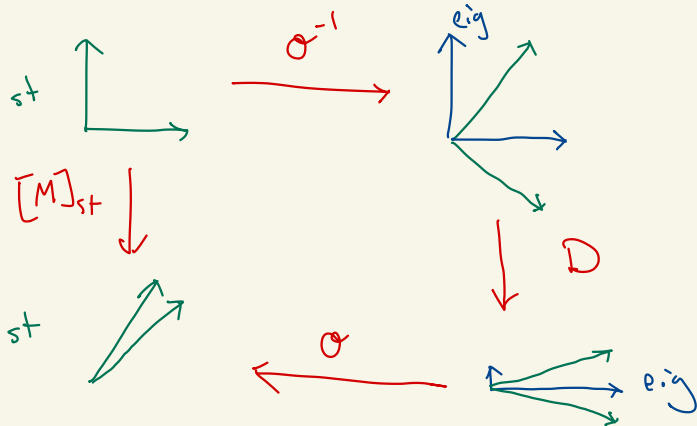
Then A has an orthonormal basis of eigenvectors,
i.e. there is an orthogonal change-of-basis matrix Q
which diagonalizes A :

$$\begin{aligned} A &= Q D Q^{-1} \\ &= Q \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} Q^{-1}. \end{aligned}$$

Complex: if $A = A^*$ (self-adjoint), then

$A = U \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} U^*$ with U unitary and
the eigenvalues λ_i are real.

Intuition: Take a linear map $M: \mathbb{R}^n \rightarrow \mathbb{R}^n$ and
find its matrix in the standard basis $[M]_{st}$.



Ex: Classical Phonons in solids (a sneak peek of Fourier Transform)

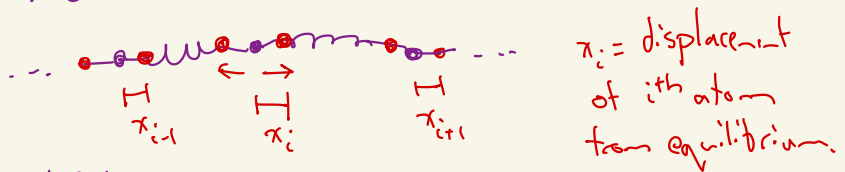
Imagine a bunch of atoms arranged in a ring:



n is very large
($n \rightarrow \infty$)

Each one interacts with its nearest neighbor (with periodic boundary conditions), modeled as a spring.

• They each oscillate around an equilibrium position:



Let $\vec{r} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$. The potential energy of the system

check: stretched (high energy) $x_{i+1} > x_i$
neutral (zero) $x_{i+1} = x_i$
compressed (high energy) $x_{i+1} < x_i$

spring constant $\downarrow$

Hook's law $\swarrow$

$$V(\vec{r}) = \sum_{i=1}^n \frac{1}{2} c (x_i - x_{i+1})^2 \quad (\text{where } x_{n+1} = x_1, x_0 = x_n)$$

So Newton's law gives

$$m \ddot{x}_j = -\frac{\partial V}{\partial x_j} = -c(x_{j-1} - x_j - (x_j - x_{j+1}))$$

$$= c(-x_{j-1} + 2x_j - x_{j+1})$$

This gives a matrix equation:

$$\frac{m}{c} \ddot{\vec{x}} = \begin{pmatrix} 2 & -1 & & -1 \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ & & \ddots & \ddots & \ddots \\ -1 & & -1 & 2 \end{pmatrix} \vec{x} \quad (*)$$

Spectral theorem: $L = L^T$, so $\exists$ orb of eigenvectors and the eigenvalues of L are real.

Actually, this matrix isn't too hard to diagonalize:

its eigenvectors $v_0, v_1, \dots, v_{n-1}$ are "plane waves":

$$v_k = \begin{pmatrix} e^{i\frac{2\pi}{n}k \cdot 1} \\ e^{i\frac{2\pi}{n}k \cdot 2} \\ \vdots \\ e^{i\frac{2\pi}{n}k(n-1)} \end{pmatrix}$$

Note: using $e^{i\theta} = \cos\theta + i\sin\theta$, we can make these real eigenvectors. But the algebra is annoying.

Check: the j^{th} entry of

$$L v_k = \lambda_k v_k$$

$$-e^{i\frac{2\pi}{n}(j-1)k} + 2e^{i\frac{2\pi}{n}jk} - e^{i\frac{2\pi}{n}j(k+1)} = \lambda_k e^{i\frac{2\pi}{n}jk}$$

$$\underbrace{\begin{pmatrix} -e^{-i\frac{2\pi}{n}k} & -e^{i\frac{2\pi}{n}k} \\ -e^{-i\frac{2\pi}{n}k} & -e^{i\frac{2\pi}{n}k} \end{pmatrix} + 2}_{= \lambda_k} e^{i\frac{2\pi}{n}jk} \leftarrow 2\cos\theta = e^{i\theta} + e^{-i\theta}$$

$$= \lambda_k = -2\cos\frac{2\pi}{n}k + 2 \leftarrow \text{eigenvalues } \lambda_k \text{ are real, as promised.}$$

Expand $x(t)$ in this eigenbasis at each time t :

$$x(t) = x_0(t)v_0 + \dots + x_{n-1}(t)v_{n-1}$$

So plugging in $Lv_k = \lambda_k v_k$ into (*),

$$\ddot{x} = \frac{c}{m} L x(t) = \frac{c}{m} (\lambda_0 x_0(t)v_0 + \lambda_1 x_1(t)v_1 + \dots + \lambda_{n-1} x_{n-1}(t)v_{n-1})$$

This diagonalization has decoupled the n equations:
for each j ,

$$\ddot{x}_j(t) = \lambda_j \frac{c}{m} x_j(t)$$

$$\Rightarrow x_j(t) = e^{\pm \sqrt{\lambda_j \frac{c}{m}} t}$$

The "+" solution is unphys. cal: these waves grow with time!

So the full solution is

$$x(t) = e^{-\sqrt{\lambda_1 \frac{c}{m}} t} v_1 + \dots + e^{-\sqrt{\lambda_n \frac{c}{m}} t} v_n$$

$$e^{-\sqrt{\lambda_1 \frac{c}{m}} t} + \dots + e^{-\sqrt{\lambda_n \frac{c}{m}} t}$$

linear combo of plane waves!

So: we should really think of the displacements $x(t)$ as a linear combo of waves:

$$v_0 = \text{—}$$

$$v_1 = \sim$$

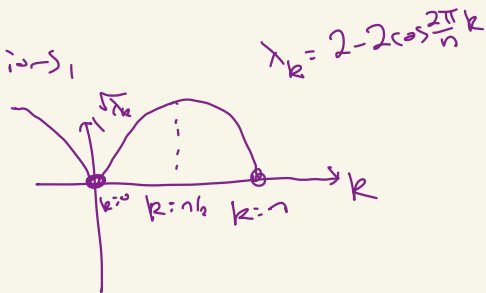
$\vdots$

$$v_k = \text{〰}$$

$$\lambda_k = 2 - 2\cos\frac{2\pi}{n}k$$

Since bigger λ_k give faster oscillations,

the $e^{-\sqrt{\lambda_k \frac{c}{m}} t}$ causes these waves to decay faster!



This will return!

Test Friday Sep 26.

Series - Linear algebra.